

# Frameworks with forced symmetry I: Reflections and rotations

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## Abstract

We give a combinatorial characterization of generic frameworks that are minimally rigid under the additional constraint of maintaining symmetry with respect to a finite order rotation or a reflection. To establish these results we develop a new technique for deriving linear representations of sparsity matroids on colored graphs and extend the direction network method of proving rigidity characterizations to handle reflections.

## 1. Introduction

A  $\Gamma$ -framework is a planar structure made of *fixed-length bars* connected by *universal joints* with full rotational freedom. Additionally, the bars and joints are symmetric with respect to the action of a point group  $\Gamma$ . The allowed motions preserve the *length* and *connectivity* of the bars and *symmetry* with respect to the same group  $\Gamma$ . This model is very similar in spirit to that of the *periodic frameworks*, introduced in [3], that have recently received a lot of attention, motivated mainly by applications to zeolites [11, 12]; see [8] for a discussion of the history.

When all the allowed motions are Euclidean isometries, a framework is *rigid* and otherwise it is *flexible*. In this paper, we give a *combinatorial* characterization of minimally rigid, generic  $\Gamma$ -frameworks when  $\Gamma$  is either a finite order rotation or a reflection. To minimize notation, we call these *cone frameworks* and *reflection frameworks* respectively. Since these  $\Gamma$  are isomorphic to some  $\mathbb{Z}/k\mathbb{Z}$  for an integer  $k \geq 2$ , we also make this identification from now on.

**1.1. The algebraic setup and combinatorial model** Formally a  $\Gamma$ -framework is given by a triple  $(\tilde{G}, \varphi, \tilde{\ell})$ , where  $\tilde{G}$  is a finite graph,  $\varphi$  is a  $\Gamma$ -action on  $\tilde{G}$  that is free on the vertices and edges, and  $\tilde{\ell} = (\ell_{ij})_{ij \in E(\tilde{G})}$  is a vector of positive *edge lengths* assigned to the edges of  $\tilde{G}$ . A *realization*  $\tilde{G}(\mathbf{p}, \Phi)$  is an assignment of points  $\mathbf{p} = (\mathbf{p}_i)_{i \in V(\tilde{G})}$  and a faithful representation of  $\Gamma$  by isometries  $\Phi : \Gamma \rightarrow \text{Euc}(2)$  such that:

$$\|\mathbf{p}_j - \mathbf{p}_i\|^2 = \ell_{ij}^2 \quad \text{for all edges } ij \in E(\tilde{G}) \quad (1)$$

$$\mathbf{p}_{\varphi(\gamma)\cdot i} = \Phi(\gamma) \cdot \mathbf{p}_i \quad \text{for all } \gamma \in \mathbb{Z}/k\mathbb{Z} \text{ and } i \in V(\tilde{G}) \quad (2)$$

The set of all realizations is defined to be the *realization space*  $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$  and its quotient by the Euclidean isometries  $\mathcal{C}(\tilde{G}, \varphi, \tilde{\ell}) = \mathcal{R}(\tilde{G}, \varphi, \tilde{\ell}) / \text{Euc}(2)$  to be the configuration space. A realization is *rigid* if it is isolated in the configuration space and otherwise *flexible*.

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As the combinatorial model for reflection frameworks it will be more convenient to use colored graphs. A *colored graph*  $(G, \gamma)$  is a finite, directed graph  $G$ , with an assignment  $\gamma = (\gamma_{ij})_{ij \in E(G)}$  of an element of a group  $\Gamma$  to each edge.

A straightforward specialization of covering space theory (see, e.g., [9, Section 9]) associates  $(\tilde{G}, \varphi)$  with a colored graph  $(G, \gamma)$ :  $G$  is the quotient of  $\tilde{G}$  by  $\Gamma$ , and the colors encode the covering map  $\tilde{G} \rightarrow G$  via a natural map  $\rho : \pi_1(G, b) \rightarrow \Gamma$ . In this setting, the choice of base vertex does not matter, and indeed, we may define  $\rho : H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z}$  and obtain the same theory. See Section 2.1 for the definition of  $\rho$  via the colors.

**1.2. Main Theorems** We can now state the main results of this paper.

**Theorem 1.** *A generic rotation framework is minimally rigid if and only if its associated colored graph is cone-Laman.*

**Theorem 2.** *A generic reflection framework is minimally rigid if and only if its associated colored graph is reflection-Laman.*

The *cone-Laman* and *reflection-Laman* graphs appearing in the statement are defined in Section 2. Genericity has its standard meaning from algebraic geometry: the set of non-generic reflection frameworks is a measure-zero algebraic set, and a small perturbation of a non-generic reflection framework yields a generic one. The cone-Laman and reflection-Laman conditions may be checked by efficient, combinatorial algorithms [2], making Theorems 1 and 2 good characterizations.

**1.3. Infinitesimal rigidity and direction networks** As in all known proofs of “Maxwell-Laman-type” theorems such as Theorems 1 and 2, we give a combinatorial characterization of a linearization of the problem known as *generic infinitesimal rigidity*. Arguments similar to those in [1] show that, generically, infinitesimal rigidity and rigidity coincide.

To characterize infinitesimal rigidity, we use a *direction network* method (cf. [8, 14, 16]). A  $\Gamma$ -*direction network*  $(\tilde{G}, \varphi, \mathbf{d})$  is a symmetric graph, along with a symmetric assignment of directions with one direction  $\mathbf{d}_{ij}$  per edge. The *realization space* of a direction network is the set of solutions  $\tilde{G}(\mathbf{p})$  to the system of equations:

$$\left\langle \mathbf{p}_j - \mathbf{p}_i, \mathbf{d}_{ij}^\perp \right\rangle = 0 \quad \text{for all edges } ij \in E(\tilde{G}) \quad (3)$$

$$\mathbf{p}_{\varphi(\gamma)\cdot i} = \Phi(\gamma) \cdot \mathbf{p}_i \quad \text{for all } \gamma \in \mathbb{Z}/k\mathbb{Z} \text{ and } i \in V(\tilde{G}) \quad (4)$$

where the  $\mathbb{Z}/k\mathbb{Z}$ -action  $\Phi$  on the plane is by either a finite order rotation through angle  $2\pi/k$  or a reflection through the  $y$ -axis. By (4), a  $\Gamma$ -direction network is determined by assigning a direction to each edge of the colored quotient graph  $(G, \gamma)$  of  $(\tilde{G}, \varphi)$  (cf. [9, Lemma 17.2]). A realization of a  $\Gamma$ -direction network is *faithful* if none of the edges of its graph have coincident endpoints and *collapsed* if all the endpoints coincide.

A basic fact in the theory of finite planar frameworks [4, 14, 16] is that, if a direction network has faithful realizations, the dimension of the realization space is equal to that of the space of infinitesimal motions of a generic framework with the same underlying graph. This is also true in the forced-symmetric case when the symmetry group contains only orientation-preserving elements, as in [8], or the finite order rotations considered here. Thus, a characterization of generic direction networks with a 3-dimensional space of faithful realizations implies a characterization of generic minimal rigidity by a straightforward sequence of steps. We will show:

**Theorem 3.** *A generic cone direction network has a faithful realization that is unique up to scale and translation if and only if its associated colored graph is cone-Laman.*

From this, Theorem 1 follows using a slight modification of the arguments in [8, Section 17-18], which we present quickly in Section 4 where we will highlight where the proof breaks down for the reflection case. The main novelty in the proof of Theorem 3 is that we make a direct geometric argument for the key Proposition 3.1, since standard results on linear representations of matroid unions don't apply to the system (3)–(4).

The situation for reflection direction networks is more complicated. The reasoning used to transition from direction networks to infinitesimal rigidity in the orientation-preserving case does *not* apply verbatim in the presence of reflections. Thus, we will need to rely on a more technical analogue of Theorem 3, which we state after giving an important definition.

Let  $(\tilde{G}, \varphi, \mathbf{d})$  be a direction network and define  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  to be the direction network with  $(\mathbf{d}^\perp)_{ij} = (\mathbf{d}_{ij})^\perp$ . These two direction networks form a *special pair* if:

- $(\tilde{G}, \varphi, \mathbf{d})$  has a, unique up to scale and translation, faithful realization.
- $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has only collapsed realizations.

**Theorem 4.** *Let  $(G, \gamma)$  be a colored graph with  $n$  vertices,  $2n - 1$  edges, and lift  $(\tilde{G}, \varphi)$ . Then there are directions  $\mathbf{d}$  such that the direction networks  $(\tilde{G}, \varphi, \mathbf{d})$  and  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  are a special pair if and only if  $(G, \gamma)$  is reflection-Laman.*

Briefly, we will use Theorem 4 as follows: the faithful realization of  $(\tilde{G}, \varphi, \mathbf{d})$  gives a symmetric immersion of the graph  $\tilde{G}$  that can be interpreted as a framework, and the fact that  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has only collapsed realizations will imply that the only symmetric infinitesimal motions of this framework correspond to translation parallel to the reflection axis.

**1.4. History and other related work** This paper is part of a sequence extending our results about periodic frameworks [8], and the results (and proofs) reported here have been previously circulated in other preprints: Theorems 1 and 3 in [9]. Theorems 2 and 4 in [10].

More recently, Jordan, Kaszanitzky, and Tanigawa have announced a generalization of the results here to dihedral groups generated by an odd number of reflections using techniques very different from ours [6]. Tanigawa [15] has recently elaborated on the techniques introduced in [8, 9] obtaining, in the process, a number of new results on scene analysis and body-bar rigidity with forced symmetry.

**1.5. Notations and terminology** In this paper, all graphs  $G = (V, E)$  may be multi-graphs. Typically, the number of vertices, edges, and connected components are denoted by  $n$ ,  $m$ , and  $c$ , respectively. The notation for a colored graph is  $(G, \gamma)$ , and a symmetric graph with a free  $\mathbb{Z}/k\mathbb{Z}$ -action is denoted by  $(\tilde{G}, \varphi)$ . If  $(\tilde{G}, \varphi)$  is the lift of  $(G, \gamma)$ , we denote the fiber over a vertex  $i \in V(G)$  by  $\tilde{i}_\gamma$ , with  $\gamma$  ranging over  $\mathbb{Z}/k\mathbb{Z}$ . The fiber over a directed edge  $ij \in E(G)$  with color  $\gamma_{ij}$  consists of the edges  $\tilde{i}_\gamma \tilde{j}_{\gamma+\gamma_{ij}}$  for  $\gamma$  ranging over  $\mathbb{Z}/k\mathbb{Z}$ .

We also use  $(k, \ell)$ -sparse graphs [7] and their generalizations. For a graph  $G$ , a  $(k, \ell)$ -basis is a maximal  $(k, \ell)$ -sparse subgraph; a  $(k, \ell)$ -circuit is an edge-wise minimal subgraph that is not  $(k, \ell)$ -sparse; and a  $(k, \ell)$ -component is a maximal subgraph that has a spanning  $(k, \ell)$ -graph.

Points in  $\mathbb{R}^2$  are denoted by  $\mathbf{p}_i = (x_i, y_i)$ , indexed sets of points by  $\mathbf{p} = (\mathbf{p}_i)$ , and direction vectors by  $\mathbf{d}$  and  $\mathbf{v}$ . For any vector  $\mathbf{v}$ , we denote its counter-clockwise rotation by  $\pi/2$  by  $\mathbf{v}^\perp$ .

Realizations of a reflection direction network  $(\tilde{G}, \varphi, \mathbf{d})$  are written as  $\tilde{G}(\mathbf{p})$ , as are realizations of abstract  $\Gamma$ -frameworks. Context will always make clear the type of realization under consideration.

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## 2. Cone- and Reflection-Laman graphs

In this section we define cone-Laman and reflection-Laman graphs and develop the properties we need. We start by recalling some general facts about colored graphs and the associated map  $\rho$ .

**2.1. The map  $\rho$  and equivalent colorings** Let  $(G, \gamma)$  be a  $\mathbb{Z}/k\mathbb{Z}$ -colored graph. Since all the colored graphs in this paper have  $\mathbb{Z}/k\mathbb{Z}$  colors, from now on we make this assumption and write simply “colored graph”.

The map  $\rho : H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z}$  is defined on oriented cycles by adding up the colors on the edges; if the cycle traverses the edge in reverse, then the edge’s color is added with a minus sign, and otherwise it is added without a sign. As the notation suggests,  $\rho$  extends to a homomorphism from  $H_1(G, \mathbb{Z})$  to  $\mathbb{Z}/k\mathbb{Z}$ , and it is well-defined even if  $G$  is not connected [8, Section 2]. The  $\rho$ -image of a colored graph is defined to be *trivial* if it contains only the identity.

We say  $\gamma$  and  $\eta$  are *equivalent colorings* of a graph  $G$  if the corresponding lifts are isomorphic covers of  $G$ . Equivalently,  $\gamma$  and  $\eta$  are equivalent if the induced representations  $H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z}$  are identical. Since all of the characteristics of rigidity or direction networks for a colored graph can be defined on the lift,  $(G, \gamma)$  and  $(G, \eta)$  have the same generic rigidity properties and the same generic rank for direction networks. We record the following lemma which will simplify the exposition of some later proofs.

**Lemma 2.1.** *Suppose  $G'$  is a subgraph of  $(G, \gamma)$  such that  $\rho(H_1(G', \mathbb{Z}))$  is trivial. Then, there is an equivalent coloring  $\eta$  such that  $\eta_{ij} = 0$  for all edges  $ij$  in  $G'$ .*

*Proof.* It suffices to construct colors  $\eta_{ij}$  such that the induced representation  $H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z}$  is equal to  $\rho$ . There is no loss of generality in assuming that  $G$  is connected. Choose a spanning forest  $T'$  of  $G'$ . Extend  $T'$  to a spanning tree  $T$  of  $G$ , and set  $\eta_{ij} = 0$  for all  $ij \in E(T)$ . Each edge  $ij$  in  $G - T$  creates a fundamental (oriented) cycle  $C_{ij}$  with respect to  $T$ . Define  $\eta_{ij} = \pm\rho(C_{ij})$  with the sign being positive if the orientation of  $ij$  and  $C_{ij}$  agree and negative otherwise. Since, by hypothesis,  $\rho(C) = 0$  for all cycles in  $G'$ , all its edges are colored 0 by  $\eta$ .  $\square$

**2.2. Cone- and Reflection-Laman graphs** Let  $(G, \gamma)$  be a colored graph with  $n$  vertices and  $m$  edges. We define  $(G, \gamma)$  to be a *cone-Laman graph* or *reflection-Laman graph* (the two families are the same) if: the number of edges  $m = 2n - 1$ , and for all subgraphs  $G'$ , spanning  $n'$  vertices,

$m'$  edges,  $c'$  connected components with non-trivial  $\rho$ -image and  $c'_0$  connected components with trivial  $\rho$ -image

$$m' \leq 2n' - c' - 3c'_0 \quad (5)$$

The underlying graph  $G$  of a cone-Laman graph is easily seen to be a  $(2, 1)$ -graph.

**2.3. Ross graphs and circuits** Another family we need is that of *Ross graphs* (see [2] for an explanation of the terminology). These are colored graphs with  $n$  vertices,  $m = 2n - 2$  edges, satisfying the sparsity counts

$$m' \leq 2n' - 2c' - 3c'_0 \quad (6)$$

using the same notations as in (5). In particular, Ross graphs  $(G, \gamma)$  have as their underlying graph, a  $(2, 2)$ -graph  $G$ , and are thus connected [7].

A *Ross-circuit*<sup>1</sup> is a colored graph that becomes a Ross graph after removing *any* edge. The underlying graph  $G$  of a Ross-circuit  $(G, \gamma)$  is a  $(2, 2)$ -circuit, and these are also known to be connected [7], so, in particular, a Ross-circuit has  $c'_0 = 0$ , and thus satisfies (5) on the whole graph. Since (5) is always at least (6), we see that every Ross-circuit is reflection-Laman. Because reflection-Laman graphs are  $(2, 1)$ -graphs and subgraphs that are  $(2, 2)$ -sparse are, in addition, Ross-sparse, we get the following structural result.

**Proposition 2.2** ([2, Lemma 11]). *Let  $(G, \gamma)$  be a reflection-Laman graph. Then each  $(2, 2)$ -component of  $G$  contains at most one Ross-circuit, and in particular, the Ross-circuits in  $(G, \gamma)$  are vertex disjoint.*

**2.4. Cone-(2, 2) graphs** A colored graph  $(G, \gamma)$  is defined to be a *cone-(2, 2)* graph, if it has  $n$  vertices,  $m = 2n$  edges, and satisfies the sparsity counts

$$m' \leq 2n' - 2c'_0 \quad (7)$$

using the same notations as in (5). The link with cone-Laman graphs is the following straightforward proposition:

**Proposition 2.3.** *A colored graph  $(G, \gamma)$  is cone-Laman if and only if, after adding a copy of any colored edge, the resulting colored graph is a cone-(2, 2) graph.*  $\square$

**2.5. Reflection-(2, 2) graphs** A colored graph  $(G, \gamma)$  is defined to be a *reflection-(2, 2)* graph, if it has  $n$  vertices,  $m = 2n - 1$  edges, and satisfies the sparsity counts

$$m' \leq 2n' - c' - 2c'_0 \quad (8)$$

using the same notations as in (5). The relationship between Ross graphs and reflection-(2, 2) graphs we will need is:

**Proposition 2.4.** *Let  $(G, \gamma)$  be a Ross-graph. Then for either*

- *an edge  $ij$  with any color where  $i \neq j$*
- *or a self-loop  $\ell$  at any vertex  $i$  colored by 1*

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<sup>1</sup>The matroid of Ross graphs has more circuits, but these are the ones we are interested in here. See Section 2.5.

the graph  $(G + ij, \gamma)$  or  $(G + \ell, \gamma)$  is reflection-(2, 2).

*Proof.* Adding  $ij$  with any color to a Ross  $(G, \gamma)$  creates either a Ross-circuit, for which  $c'_0 = 0$  or a Laman-circuit with trivial  $\rho$ -image. Both of these types of graph meet this count, and so the whole of  $(G + ij, \gamma)$  does as well.  $\square$

It is easy to see that every reflection-Laman graph is a reflection-(2, 2) graph. The converse is not true.

**Proposition 2.5.** *A colored graph  $(G, \gamma)$  is a reflection-Laman graph if and only if it is a reflection-(2, 2) graph and no subgraph with trivial  $\rho$ -image is a (2, 2)-block.*  $\square$

Let  $(G, \gamma)$  be a reflection-Laman graph, and let  $G_1, G_2, \dots, G_t$  be the Ross-circuits in  $(G, \gamma)$ . Define the *reduced graph*  $(G^*, \gamma)$  of  $(G, \gamma)$  to be the colored graph obtained by contracting each  $G_i$ , which is not already a single vertex with a self-loop (this is necessarily colored 1), into a new vertex  $v_i$ , removing any self-loops created in the process, and then adding a new self-loop with color 1 to each of the  $v_i$ . By Proposition 2.2 the reduced graph is well-defined.

**Proposition 2.6.** *Let  $(G, \gamma)$  be a reflection-Laman graph. Then its reduced graph is a reflection-(2, 2) graph.*

*Proof.* Let  $(G, \gamma)$  be a reflection-Laman graph with  $t$  Ross-circuits with vertex sets  $V_1, \dots, V_t$ . By Proposition 2.2, the  $V_i$  are all disjoint. Now select a Ross-basis  $(G', \gamma)$  of  $(G, \gamma)$ . The graph  $G'$  is also a (2, 2)-basis of  $G$ , with  $2n - 1 - t$  edges, and each of the  $V_i$  spans a (2, 2)-block in  $G'$ . The  $(k, \ell)$ -sparse graph Structure Theorem [7, Theorem 5] implies that contracting each of the  $V_i$  into a new vertex  $v_i$  and discarding any self-loops created, yields a (2, 2)-sparse graph  $G^+$  on  $n^+$  vertices and  $2n^+ - 1 - t$  edges. It is then easy to check that adding a self-loop colored 1 at each of the  $v_i$  produces a colored graph satisfying the reflection-(2, 2) counts (8) with exactly  $2n^+ - 1$  edges. Since this is the reduced graph, we are done.  $\square$

**2.6. Decomposition characterizations** A *map-graph* is a graph with exactly one cycle per connected component. A *cone-(1, 1)* or *reflection-(1, 1)* graph is defined to be a colored graph  $(G, \gamma)$  where  $G$ , taken as an undirected graph, is a map-graph and the  $\rho$ -image of each connected component is non-trivial.

**Lemma 2.7.** *Let  $(G, \gamma)$  be a colored graph. Then  $(G, \gamma)$  is a reflection-(2, 2) graph if and only if it is the union of a spanning tree and a reflection-(1, 1) graph.*

*Proof.* By [18, Matroid Theorem], reflection-(1, 1) graphs are equivalent to graphs satisfying

$$m' \leq n' - c'_0 \tag{9}$$

for every subgraph  $G'$ . Thus, (8) is

$$m' \leq (n' - c'_0) + (n' - c' - c'_0) \tag{10}$$

The second term in (10) is well-known to be the rank function of the graphic matroid, and the Lemma follows from the Edmonds-Rota construction [5] and the Matroid Union Theorem.  $\square$

Nearly the same proof yields the analogous statement for cone-(2, 2) graphs.

**Proposition 2.8.** A colored graph  $(G, \gamma)$  is cone-(2, 2) if and only if it is the union of two cone-(1, 1) graphs.  $\square$

In the sequel, it will be convenient to use this slight refinement of Lemma 2.7.

**Proposition 2.9.** Let  $(G, \gamma)$  be a reflection-(2, 2) graph. Then there is an equivalent coloring  $\gamma'$  of the edges of  $G$  such that the tree in the decomposition as in Lemma 2.7 has all edges colored by the identity.

*Proof.* Apply Lemma 2.1 to the tree in the decomposition.  $\square$

Proposition 2.9 has the following re-interpretation in terms of the symmetric lift  $(\tilde{G}, \varphi)$ :

**Proposition 2.10.** Let  $(G, \gamma)$  be a reflection-(2, 2) graph. Then for a decomposition, as provided by Proposition 2.9, into a spanning tree  $T$  and a reflection-(1, 1) graph  $X$ :

- Every edge  $ij \in T$  lifts to the two edges  $\tilde{i}_0\tilde{j}_0$  and  $\tilde{i}_1\tilde{j}_1$ . In particular, the vertex representatives in the lift all lie in a single connected component of the lift of  $T$ .
- Each connected component of  $X$  lifts to a connected graph.

**2.7. The overlap graph of a cone-(2, 2) graph** Let  $(G, \gamma)$  be cone-(2, 2) and fix a decomposition of it into two cone-(1, 1) graphs  $X$  and  $Y$ . Let  $X_i$  and  $Y_i$  be the connected components of  $X$  and  $Y$ , respectively. Also select a base vertex  $x_i$  and  $y_i$  for each connected component of  $X$  and  $Y$ , with all base vertices on the cycle of their component. Denote the collection of base vertices by  $B$ . We define the *overlap graph* of  $(G, X, Y, B)$  to be the directed graph with:

- Vertex set  $B$ .
- A directed edge from  $x_i$  to  $y_j$  if  $y_j$  is a vertex in  $X_i$ .
- A directed edge from  $y_j$  to  $x_i$  if  $x_i$  is a vertex in  $Y_j$ .

The property of the overlap graph we need is:

**Proposition 2.11.** Let  $(G, \gamma)$  be a cone-(2, 2) graph with fixed decomposition and choice of base vertices. The overlap graph of  $(G, Y, R, B)$  has a directed cycle in each connected component.

*Proof.* Every vertex has exactly one incoming edge, since each vertex is in exactly one connected component of each of  $X$  and  $Y$ . Directed graphs with an in-degree exactly one orientation have exactly one directed cycle per connected component (see, e.g., [13]).  $\square$

### 3. Cone direction networks

In this section, we prove Theorem 3. The main step in the proof is:

**Proposition 3.1.** Let  $(G, \gamma)$  be a cone-(2, 2) graph. Then every realization of a generic direction network on  $(G, \gamma)$  is collapsed, with all vertices placed on the rotation center.

Next, we introduce *colored direction networks*, which will be convenient to work with.

**3.1. The colored realization system** The system of equations (3)–(4) defining the realization space of a reflection direction network  $(\tilde{G}, \varphi, \mathbf{d})$  is linear, and as such has a well-defined dimension. Let  $(G, \gamma)$  be the colored quotient graph of  $(\tilde{G}, \varphi)$ .

Since our setting is symmetric frameworks, we require that the assigned directions also be symmetric. In other words, if  $\mathbf{d}_{ij}$  is the direction for the edge  $\tilde{i}_0\tilde{j}_{\gamma_{ij}}$  in the fiber of  $ij$ , then  $R_k^\gamma \mathbf{d}_{ij}$  is the direction for the edge  $\tilde{i}_k\tilde{j}_{\gamma+\gamma_{ij}}$  where  $R_k$  is the counter-clockwise rotation of order  $k$  here and throughout this section. Thus, to specify a direction network on  $(\tilde{G}, \varphi)$ , we need only assign a direction to one edge in each edge orbit. Furthermore, since the directions and realizations must be symmetric, the system can be reduced to the following one where the unknowns consist of  $n$  points  $\mathbf{p}_i$ , one for each vertex  $i$  of the quotient graph where  $\mathbf{d}_{ij} = \mathbf{d}_{\tilde{i}_0\tilde{j}_{\gamma_{ij}}}$ .

$$\langle R_k^{\gamma_{ij}} \cdot \mathbf{p}_j - \mathbf{p}_i, \mathbf{d}_{ij} \rangle = 0 \quad \text{for all edges } ij \in E(G) \quad (11)$$

Proposition 3.1 can be reinterpreted to say that the colored realization system (11), in matrix form, is a linear representation for the matroid of cone-(2, 2) graphs. This representation is different than the ones obtained via standard results on matroid union.

**3.2. Genericity** Let  $(G, \gamma)$  be a colored graph with  $m$  edges. A statement about direction networks  $(\tilde{G}, \varphi, \mathbf{d})$  is *generic* if it holds on the complement of a proper algebraic subset of the possible direction assignments, which is canonically identified with  $\mathbb{R}^{2m}$ . Some facts about generic statements that we use frequently are:

- Almost all direction assignments are generic.
- If a set of directions is generic, then so are all sufficiently small perturbations of it.
- If two properties are generic, then their intersection is as well.
- The maximum rank of (11) is a generic property.

This next proof is relatively standard.

**3.3. Proof that Proposition 3.1 implies Theorem 3** We prove each direction of the statement in turn. Since it is technically easier, we prove the equivalent statement on colored direction networks.

**Cone-laman graphs generically have faithful realizations** The proof in [8, Section 15.3] applies with small modifications.

**Cone-laman circuits have collapsed edges** For the other direction, we suppose that  $(G, \gamma)$  has  $n$  vertices and is not cone-Laman. If the number of edges  $m$  is less than  $2n - 1$ , then the realization space of any direction network is at least 2-dimensional, so it contains more than just rescalings. Thus, we assume that  $m \geq 2n - 1$ . In this case,  $G$  is not cone-Laman sparse, so it contains a cone-Laman circuit  $(G', \gamma)$ . Thus, we are reduced to showing that any cone-Laman circuit has, generically, only realizations with collapsed edges, since these then force collapsed edges in any realization of a generic colored direction network on  $(G, \gamma)$ .

There are two types of subgraphs  $(G', \gamma)$  that constitute minimal violations of cone-Laman sparsity:

- $(G', \gamma)$  is a cone-(2, 2) graph.
- $(G', \gamma)$  has trivial  $\rho$ -image, and is a (2, 2)-block.

If  $(G', \gamma)$  is a cone-(2, 2) graph, then Proposition 3.1 applies to it, and we are done.

For the other type, we may assume that the colors on  $G'$  are zero by Lemma 2.1 in which case the system (11) on  $(G', \gamma)$  is equivalent to a direction network on  $G'$  as a finite, unsymmetric, uncolored graph. Thus, the Parallel Redrawing Theorem [17, Theorem 4.1.4] in the form [14, Theorem 3] applies directly to show that all realizations of  $G'$  have only collapsed edges.  $\square$

The rest of this section proves Proposition 3.1.

**3.4. Geometry of some generic linear projections** We first establish some geometric results we need below.

#### 3.4.1 Affine lines

Given a unit vector  $\mathbf{v} \in \mathbb{R}^2$  and a point  $\mathbf{p} \in \mathbb{R}^2$ , we denote by  $\ell(\mathbf{v}, \mathbf{p})$  the affine line consisting of points  $\mathbf{q}$  where

$$\mathbf{q} - \mathbf{p} = \lambda \mathbf{v}$$

for some scalar  $\lambda \in \mathbb{R}$ . I.e.  $\ell(\mathbf{v}, \mathbf{p})$  is the line through  $\mathbf{p}$  in direction  $\mathbf{v}$ .

#### 3.4.2 An important linear equation

The following is a key lemma which will determine where certain points must lie when solving a cone direction network.

**Lemma 3.2.** *Suppose  $R$  is a non-trivial rotation about the origin,  $\mathbf{v}^*$  is a unit vector and  $\mathbf{p}$  satisfies*

$$(R - I)\mathbf{p} = \lambda \mathbf{v}^*$$

*for some  $\lambda \in \mathbb{R}$ . Then, for some  $C \in \mathbb{R}$ , we have  $\mathbf{p} = C\mathbf{v}$  where  $\mathbf{v} = R_{\pi/2}R^{-1/2}\mathbf{v}^*$ ,  $R^{-1/2}$  is the inverse of a square root of  $R$ , and  $R_{\pi/2}$  is the counter-clockwise rotation through angle  $\pi/2$ .*

*Proof.* A computation shows that  $(R - I)R^{-1/2} = R^{1/2} - R^{-1/2}$  is a scalar multiple of  $R_{\pi/2}$ . The Lemma follows.  $\square$

#### 3.4.3 The projection $T(\mathbf{v}, \mathbf{w}, R)$

Let  $\mathbf{v}$  and  $\mathbf{w}$  be unit vectors in  $\mathbb{R}^2$ , and  $R$  some nontrivial rotation. Denote by  $\mathbf{v}^*$  the vector  $R^{1/2} \cdot \mathbf{v}^\perp$  for some choice of square root  $R^{1/2}$  of  $R$ .

We define  $T(\mathbf{v}, \mathbf{w}, R)$  to be the linear projection from  $\ell(\mathbf{v}, 0)$  to  $\ell(\mathbf{w}, 0)$  in the direction  $\mathbf{v}^*$ . The following properties of  $T(\mathbf{v}, \mathbf{w}, R)$  are straightforward.

**Lemma 3.3.** *Let  $\mathbf{v}$  and  $\mathbf{w}$  be unit vectors, and  $R$  a nontrivial rotation. Then, the linear map  $T(\mathbf{v}, \mathbf{w}, R)$ :*

- Is defined if  $\mathbf{v}^*$  is not in the same direction as  $\mathbf{w}$ .
- Is identically zero if and only if  $\mathbf{v}^*$  and  $\mathbf{v}$  are collinear.

### 3.4.4 The scale factor of $T(\mathbf{v}, \mathbf{w}, R)$

The image  $T(\mathbf{v}, \mathbf{w}, R) \cdot \mathbf{v}$  is equal to  $\lambda \mathbf{w}$ , for some scalar  $\lambda$ . We define the *scale factor*  $\lambda(\mathbf{v}, \mathbf{w}, R)$  to be this  $\lambda$ . We need the following elementary fact about the scaling factor of  $T(\mathbf{v}, \mathbf{w}, R)$ .

**Lemma 3.4.** *Let  $\mathbf{v}$  and  $\mathbf{w}$  be unit vectors such that  $\mathbf{v}^*$  and  $\mathbf{w}$  are linearly independent. Then the scaling factor  $\lambda(\mathbf{v}, \mathbf{w}, R)$  of the linear map  $T(\mathbf{v}, \mathbf{w}, R)$  is given by*

$$\frac{\langle \mathbf{v}, (\mathbf{v}^*)^\perp \rangle}{\langle \mathbf{w}, (\mathbf{v}^*)^\perp \rangle}$$

*Proof.* The map  $T(\mathbf{v}, \mathbf{w}, R)$  is equivalent to the composition of:

- perpendicular projection from  $\ell(\mathbf{v}, 0)$  to  $\ell((\mathbf{v}^*)^\perp, 0)$ , followed by
- the inverse of perpendicular projection  $\ell(\mathbf{w}, 0) \rightarrow \ell((\mathbf{v}^*)^\perp, 0)$ .

The first map scales the length of vectors by  $\langle \mathbf{v}, (\mathbf{v}^*)^\perp \rangle$  and the second by the reciprocal of  $\langle \mathbf{w}, (\mathbf{v}^*)^\perp \rangle$ .  $\square$

From Lemma 3.4 it is immediate that

**Lemma 3.5.** *The scaling factor  $\lambda(\mathbf{v}, \mathbf{w}, R)$  is identically 0 precisely when  $R$  is an order two rotation. If  $R$  is not an order 2 rotation, then  $\lambda(\mathbf{v}, \mathbf{w}, R)$  approaches infinity as  $\mathbf{v}^*$  approaches  $\pm \mathbf{w}$ .*

### 3.4.5 Generic sequences of the map $T(\mathbf{v}, \mathbf{w}, R)$

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be unit vectors, and  $S_1, S_2, \dots, S_n$  be rotations of the form  $R_k^i$  where  $R_k$  is a rotation of order  $k$ . We define the linear map  $T(\mathbf{v}_1, S_1, \mathbf{v}_2, S_2, \dots, \mathbf{v}_n, S_n)$  to be

$$T(\mathbf{v}_1, S_1, \mathbf{v}_2, S_2, \dots, \mathbf{v}_n, S_n) = T(\mathbf{v}_n, \mathbf{v}_1, S_n) \circ T(\mathbf{v}_{n-1}, \mathbf{v}_n, S_{n-1}) \circ \cdots \circ T(\mathbf{v}_1, \mathbf{v}_2, S_1)$$

This next proposition plays a key role in the next section, where it is interpreted as providing a genericity condition for cone direction networks.

**Proposition 3.6.** *Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be pairwise linearly independent unit vectors, and  $S_1, S_2, \dots, S_n$  be rotations of the form  $R_k^i$ . Then if the  $\mathbf{v}_i$  avoid a proper algebraic subset of  $(\mathbb{S}^1)^n$  (that depends on the  $S_i$ ), the map  $T(\mathbf{v}_1, S_1, \mathbf{v}_2, S_2, \dots, \mathbf{v}_n, S_n)$  scales the length of vectors by a factor of  $\lambda \neq 1$ .*

*Proof.* If any of the  $T(\mathbf{v}_i, \mathbf{v}_{i+1}, S_i)$  are identically zero, we are done, so we assume none of them are. The map  $T(\mathbf{v}_1, S_1, \mathbf{v}_2, S_2, \dots, \mathbf{v}_n, S_n)$  then scales vectors by a factor of:

$$\lambda(\mathbf{v}_1, \mathbf{v}_2, S_1) \cdot \lambda(\mathbf{v}_2, \mathbf{v}_3, S_2) \cdots \lambda(\mathbf{v}_{n-1}, \mathbf{v}_n, S_{n-1}) \cdot \lambda(\mathbf{v}_n, \mathbf{v}_1, S_n)$$

which we denote  $\lambda$ . That  $\lambda$  is constantly one is a polynomial statement in the  $\mathbf{v}_i$  by Lemma 3.4, and so it is either always true or holds only on a proper algebraic subset of  $(\mathbb{S}^1)^n$ . This means it suffices to prove that there is one selection for the  $\mathbf{v}_i$  where  $\lambda \neq 1$ .

Without loss of generality, we may assume that  $T(\mathbf{v}_j, \mathbf{v}_{j+1}, S_j)$  is defined for all  $j$ . Note that  $|\lambda(\mathbf{v}_n, \mathbf{v}_1, S_n)|$  attains an absolute (non-zero) minimum when  $\mathbf{v}_n^* = \mathbf{v}_1^\perp$  by Lemma 3.4 and  $|\lambda(\mathbf{v}_1, \mathbf{v}_2, S_1)|$  grows arbitrary large as  $\mathbf{v}_1^*$  moves towards  $\mathbf{v}_2$ . Thus  $|\lambda|$  grows arbitrary large as  $\mathbf{v}_1^*$  moves towards  $\mathbf{v}_2$  and so is not constantly 1.  $\square$

**3.5. Proof of Proposition 3.1** It will suffice to prove the existence of a single assignment of directions  $\mathbf{d}$  to a cone-(2, 2) graph  $(G, \gamma)$  for which we can show all the realizations  $G(\mathbf{p})$  of the resulting direction network are collapsed. The generic statement is then immediate. The strategy is to first decompose  $(G, \gamma)$  into cone-(1, 1) graphs using Proposition 2.8, assign directions to each connected cone-(1, 1) graph which force a certain local geometric structure on  $G(\mathbf{p})$  (Section 3.5.1), and then to show that the properties of the overlap graph imply the whole realization must be collapsed (Sections 3.5.2–3.5.3).

### 3.5.1 Special direction networks on connected map-graphs

Let  $(G, \gamma)$  be a  $\mathbb{Z}/k\mathbb{Z}$ -colored graph that is a connected cone-(1, 1) graph. We select and fix a base vertex  $b \in V(G)$  that is on the (unique) cycle in  $G$ . This next lemma provides the main “gadget” that we use in the proof of Proposition 3.1 below.

**Lemma 3.7.** *Let  $k \geq 2$ , and let  $(G, \gamma)$  be a  $\mathbb{Z}/k\mathbb{Z}$ -colored graph that is a connected cone-(1, 1) graph with a base vertex  $b$ . Let  $\gamma \in \mathbb{Z}/k\mathbb{Z}$  be the  $\rho$ -image of the cycle in  $G$ , let  $\mathbf{v}$  be a unit vector, and let  $\mathbf{v}^* = (R_k^{\gamma/2} \cdot \mathbf{v})^\perp$ . We can assign directions  $\mathbf{d}$  to the edges of  $G$  so that, in all realizations of the corresponding direction network  $(\tilde{G}, \varphi, \mathbf{d})$  on the lift:*

- For each  $\gamma' \in \mathbb{Z}/k\mathbb{Z}$ , the point  $\mathbf{p}_{\tilde{b}_{\gamma'}}$  lies on the line  $\ell(R_k^{\gamma'} \cdot \mathbf{v}, 0)$
- The rest of the points all lie on the lines  $\ell(R_k^{\gamma'} \mathbf{v}^*, \mathbf{p}_{\tilde{b}_{\gamma'}})$  as  $\gamma'$  ranges over  $\mathbb{Z}/k\mathbb{Z}$ .

*Proof.* We assign directions in the lift  $\tilde{G}$  of  $G$ . We start by selecting an edge  $bi \in E(G)$  that is incident on the base vertex  $b$  and in the cycle in  $G$ . Then,  $G - bi$  is a spanning tree  $T$  of  $G$ .

Since  $T$  is contractible, it lifts to  $k$  disjoint copies of itself in  $\tilde{G}$ . Let  $\tilde{T}$  be the copy containing  $\tilde{b}_0$ . Note that  $\tilde{T}$  hits the fiber over every edge in  $G$  except for  $bi$  exactly one time and the fiber over every vertex exactly one time.

Recall that we only need to assign a direction to one edge in each orbit. We first assign every edge in  $\tilde{T}$  the direction  $\mathbf{v}^* = (R_k^{\gamma/2} \cdot \mathbf{v})^\perp$ . From this choice of directions already, it now follows by the connectivity of  $T$  that in any realization of the cone direction network induced on  $\tilde{G} - \pi^{-1}(bi)$  any point lies on some  $\ell(R_k^{\gamma'} \mathbf{v}^*, \mathbf{p}_{\tilde{b}_{\gamma'}})$ .

It only remains to specify a direction for some edge in the fiber of  $bi$ . Select the edge in the fiber over  $bi$  incident on the copy of  $i$  in  $\tilde{T}$ . Assign this edge the direction  $\mathbf{v}^*$  as well. Note that this edge is necessarily incident on the vertex  $\tilde{b}_\gamma$  since the cycle has  $\rho$ -image  $\gamma$ . The choice of direction on this edge implies that

$$\mathbf{p}_{\tilde{b}_\gamma} - \mathbf{p}_{\tilde{b}_0} = R_k^\gamma \mathbf{p}_{\tilde{b}_0} - \mathbf{p}_{\tilde{b}_0} = \lambda \mathbf{v}^*$$

for some scalar  $\lambda$ . It now follows from Lemma 3.2, applied to the rotation  $R_k^\gamma$ , that  $\mathbf{p}_{\tilde{b}_0}$  lies on  $\ell(\mathbf{v}, 0)$ .  $\square$

### 3.5.2 Proof of Proposition 3.1 for order 2 rotations

Decompose the cone-(2, 2) graph  $(G, \gamma)$  into two edge-disjoint (but not necessarily connected) cone-(1, 1) graphs  $X$  and  $Y$ , using Proposition 2.8. The order of the  $\rho$ -image of any cycle in either  $X$  or  $Y$  is always 2, so the construction of Lemma 3.7 implies that, by assigning the same

direction  $\mathbf{v}$  to every edge in  $X$ , every vertex in any realization lies on a single line through the origin in the direction of  $\mathbf{v}$ . Similarly for edges in  $Y$  assigned a direction  $\mathbf{w}$  different than  $\mathbf{v}$ .

Since every vertex is at the intersection of two skew lines through the origin, the proposition is proved.  $\square$

### 3.5.3 Proof of Proposition 3.1 for rotations of order $k \geq 3$

Using Proposition 2.8, fix a decomposition of the cone-(2,2) graph  $(G, \gamma)$  into two cone-(1,1) graphs  $X$  and  $Y$  and choose base vertices for the connected components of  $X$  and  $Y$ . This determines an overlap graph  $D$ . Since we deal with them independently of  $X$  and  $Y$ , we define  $G_i$  to be the connected components of  $X$  and  $Y$ , and recall that these partition the edges of  $G$ . We denote the base vertex of  $G_i$  by  $b_i$ . Since we use subscripts to denote the subgraph of the base vertex, in this section we will use the notation  $\gamma \cdot \tilde{b}_i$  as  $\gamma$  ranges over  $\mathbb{Z}/k\mathbb{Z}$  for the fiber over vertex  $b_i$ .

**Assigning directions** For each  $G_i$ , select a unit vector  $\mathbf{v}_i$  such that:

- For any  $i, j$ , we have  $\mathbf{v}_i \neq R_k^\gamma \mathbf{v}_j$  for all  $\gamma \in \mathbb{Z}/k\mathbb{Z}$ .
- For all choices  $k_i$ , the vectors  $\mathbf{w}_i = R_k^{k_i} \mathbf{v}_i$  are generic in the sense of Proposition 3.6.

Now, for each  $G_i$  we assign directions as prescribed by Lemma 3.7 where  $\mathbf{v}_i$  is the vector input into the lemma. This is well-defined, since the  $G_i$  partition the edges of  $G$ . (They clearly overlap on the vertices—we will exploit this fact below—but it does not prevent us from assigning edge directions independently.)

We define the resulting colored direction network to be  $(G, \gamma, \mathbf{d})$  and the lifted cone direction network  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$ . We also define, as a convenience, the rotation  $S_i$  to be  $R_k^\gamma$  where  $\gamma$  is the  $\rho$ -image of the unique cycle in  $G_i$ .

**Local structure of realizations** Let  $G_i$  and  $G_j$  be distinct connected cone-(1,1) components and suppose that there is a directed edge  $b_i b_j$  in the overlap graph  $D$ . We have the following relationship between  $\mathbf{p}_{\tilde{b}_i}$  and  $\mathbf{p}_{\tilde{b}_j}$  in realizations of  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$ .

**Lemma 3.8.** *Let  $\tilde{G}(\mathbf{p})$  be a realization of the cone direction network  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$  defined above. Let vertices  $b_i$  and  $b_j$  in  $V(G)$  be the base vertices of  $G_i$  and  $G_j$ , and suppose that  $b_i b_j$  is a directed edge in the overlap graph  $D$ . Let  $\gamma \cdot \tilde{b}_i$  be some vertex in the fiber of  $b_i$ . Then for some  $\gamma'$ , we have  $\mathbf{p}_{\gamma' \cdot \tilde{b}_j} = T(R_k^{\gamma'} \mathbf{v}_i, R_k^\gamma \mathbf{v}_j, S_i) \cdot \mathbf{p}_{\gamma \cdot \tilde{b}_i}$*

The proof is illustrated in Figure 1.

*Proof.* By Lemma 3.7, the vertex  $\mathbf{p}_{\gamma \cdot \tilde{b}_i}$  lies on the line  $\ell(R_k^\gamma \mathbf{v}_i, 0)$ , and  $\mathbf{p}_{\gamma' \cdot \tilde{b}_j}$  lies on  $\ell(R_k^{\gamma'} \mathbf{v}_j^*, \mathbf{p}_{\gamma \cdot \tilde{b}_i})$  for some  $\gamma' \in \mathbb{Z}/k\mathbb{Z}$  since the vertex  $b_j$  lies in the map-graph  $G_i$ . By Lemma 3.7 applied to  $G_j$ , the vertex  $\mathbf{p}_{\gamma' \cdot \tilde{b}_j}$  lies on the line  $\ell(R_k^{\gamma'} \mathbf{v}_j, 0)$ . This is exactly the situation captured by the map  $T(R_k^{\gamma'} \mathbf{v}_i, R_k^\gamma \mathbf{v}_j, S_i)$ .  $\square$

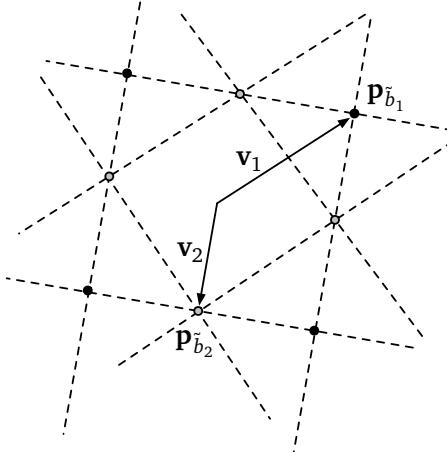


Figure 1: Example of the local structure of the proof of Proposition 3.1; in this example,  $R_4^2 \mathbf{p}_{\tilde{b}_2}$  is the image of  $p_{\tilde{b}_1}$  via the projection  $T(\mathbf{v}_1, R_4^2 \mathbf{v}_2, S_1)$ .

**Base vertices on cycles in  $D$  must be at the origin** Let  $b_i$  be the base vertex in  $G_i$  that is also on a directed cycle in  $D$ . The next step in the proof is to show that all representatives in  $b_i$  must be mapped to the origin in any realization of  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$ .

**Lemma 3.9.** *Let  $\tilde{G}(\mathbf{p})$  be a realization of the cone direction network  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$  defined above, and let  $b_i \in V(G)$  be a base vertex that is also on a directed cycle in  $D$  (one exists by Proposition 2.11). Then all vertices in the fiber over  $b_i$  must be mapped to the origin.*

*Proof.* Iterated application of Lemma 3.8 along the cycle in  $D$  containing  $b_i$  tells us that any vertex in the fiber over  $b_i$  is related to another vertex in the same fiber by a linear map meeting the hypothesis of Proposition 3.6. This implies that if any vertex in the fiber over  $b_i$  is mapped to a point not the origin, some other vertex in the same fiber is mapped to a point at a different distance to the origin. This is a contradiction, since all realizations  $\tilde{G}(\mathbf{p})$  are symmetric with respect to  $R_k$ , so, in fact the fiber over  $b_i$  was mapped to the origin.  $\square$

**All base vertices must be at the origin** So far we have shown that every base vertex  $b_i$  that is on a directed cycle in the overlap graph  $D$  is mapped to the origin in any realization  $\tilde{G}(\mathbf{p})$  of  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$ . However, since every base vertex is connected to the cycle in its connected component by a directed path in  $D$ , we can show all the base vertices are at the origin.

**Lemma 3.10.** *Let  $\tilde{G}(\mathbf{p})$  be a realization of the cone direction network  $(\tilde{G}, \varphi, \tilde{\mathbf{d}})$  defined above. Then all vertices in the fiber over  $b_i$  must be mapped to the origin.*

*Proof.* The statement is already proved for base vertices on a directed cycle in Lemma 3.9. Any base vertex not on a directed cycle, say  $b_i$ , is at the end of a directed path which starts at a vertex on the directed cycle. Thus  $\mathbf{p}_{\gamma \cdot b_i}$  is the image of 0 under some linear map, and hence is at the origin.  $\square$

**All vertices must be at the origin** The proof of Proposition 3.1 then follows from the observation that, if all the base vertices  $b_i$  must be mapped to the origin in  $\tilde{G}(\mathbf{p})$ , then Lemma 3.7

implies that every vertex in the lift of  $G_i$  lies on a family of  $k$  lines intersecting at the origin. Since every vertex is in the span of two of the  $G_i$ , and these families of lines intersect only at the origin, we are done:  $\tilde{G}(\mathbf{p})$  must put all the points at the origin.  $\square$

#### 4. Infinitesimal rigidity of cone frameworks

An argument similar to that in [1] shows that generic cone frameworks are rigid if and only if they are infinitesimally rigid. Due to symmetry, the generic rigidity of a cone framework  $(\tilde{G}, \varphi, \tilde{\ell})$  is a property of the underlying colored graph. In fact, we can identify the realization space with the following system defined on the associated colored graph  $(G, \gamma)$

$$\|R_k^{\gamma_{ij}} \mathbf{p}_j - \mathbf{p}_i\|^2 = \ell_{ij}^2 \quad \text{for all edges } ij \in E(G) \quad (12)$$

Here  $\ell_{ij}$  is equal to the length of any lift of the edge  $ij$  which, by symmetry, is independent of the lift. We think of this as a framework  $(G, \gamma, \ell)$  on the quotient graph. Via a straightforward computation we see that a vector  $\mathbf{q} \in \mathbb{R}^{2n}$  is an infinitesimal motion if and only if

$$\langle R_k^{\gamma_{ij}} \mathbf{q}_j - \mathbf{q}_i, R_k^{\gamma_{ij}} \mathbf{p}_j - \mathbf{p}_i \rangle = 0 \quad \text{for all edges } ij \in E(G) \quad (13)$$

We define  $(G, \gamma, \ell)$  to be *infinitesimally rigid* if the system (13) has rank  $2n - 1$ .

**4.1. Proof of Theorem 1** We prove necessity by inspecting (13) and then sufficiency with Theorem 3.

**The Maxwell direction** Suppose that  $(G, \gamma)$  is a colored graph on  $m = 2n - 1$  edges that is not cone-Laman-sparse. Thus  $G$  contains (at least) one of two possible types of cone-Laman-circuits which we call  $G'$ : a cone-(2, 2) graph or a (2, 2)-graph with trivial  $\rho$ -image. In the former case, the subgraph has  $n$  vertices and  $2n$  edges, and since any framework has a trivial motion arising from rotation, the system (13) has a dependency.

In the latter case, by Lemma 2.1, we may assume the edges are colored by 0, in which case the system (13) is identical to the well-known rigidity matrix for finite frameworks. Since  $G'$  is not (2, 3)-sparse, the Maxwell-Laman Theorem provides a dependency in (13).  $\square$

**The Laman direction** Theorem 3 implies that (11) has, generically, rank  $2n - 1$  if  $(G, \gamma)$  is cone-Laman. This next proposition says that (13) has the same rank.

**Proposition 4.1.** *Let  $G(\mathbf{p})$  be a realization of a cone framework with colored graph  $(G, \gamma)$ . If  $G(\mathbf{p})$  is faithful and solves the direction network  $(G, \gamma, \mathbf{d})$ , then the system (13) for  $G(\mathbf{p})$  has the same rank as (11) for  $(G, \gamma, \mathbf{d})$ .*

*Proof.* Let  $R_{\pi/2}$  be the counter-clockwise rotation through angle  $\pi/2$ . If  $G(\mathbf{p})$  faithfully solves  $(G, \gamma, \mathbf{d})$ , then  $R_k^{\gamma_{ij}} \mathbf{p}_j - \mathbf{p}_i = \alpha \mathbf{d}_{ij}$ . A vector  $\mathbf{q}$  is therefore an infinitesimal motion if and only if  $\mathbf{q}^\perp$  is a solution to (11) by this computation:

$$\begin{aligned} \langle R_k^{\gamma_{ij}} \mathbf{q}_j - \mathbf{q}_i, \mathbf{d}_{ij} \rangle &= 0 \iff \\ \langle R_{\pi/2}(R_k^{\gamma_{ij}} \mathbf{q}_j - \mathbf{q}_i), R_{\pi/2} \mathbf{d}_{ij} \rangle &= 0 \iff \\ \langle R_k^{\gamma_{ij}} R_{\pi/2} \mathbf{q}_j - R_{\pi/2} \mathbf{q}_i, R_{\pi/2} \mathbf{d}_{ij} \rangle &= 0 \end{aligned}$$

Thus, (13) and (11) have isomorphic solution spaces and hence the same rank.  $\square$

**Remark** That  $R_{\pi/2}$  and  $R_k^{\gamma_{ij}}$  commute is critical in the computation used to prove Proposition 4.1. The corresponding argument for reflection frameworks would fail since  $R_k^{\gamma_{ij}}$  would be replaced by a reflection which does not commute with  $R_{\pi/2}$ . In fact, as we will see, the ranks of the two systems can be different in the reflection case.

## 5. Special pairs of reflection direction networks

We recall, from the introduction, that for reflection direction networks,  $\mathbb{Z}/2\mathbb{Z}$  acts on the plane by reflection through the  $y$ -axis, and in the rest of this section  $\Phi(\gamma)$  refers to this action.

**5.1. Direction networks on Ross graphs** We first characterize the colored graphs for which generic direction networks have strongly faithful realizations. A realization is *strongly faithful* if no two vertices lie on top of each other. This is a stronger condition than simply being faithful which only requires that edges not be collapsed.

**Proposition 5.1.** *A generic direction network  $(\tilde{G}, \varphi, \mathbf{d})$  has a unique, up to translation and scaling, strongly faithful realization if and only if its associated colored graph is a Ross graph.*

To prove Proposition 5.1 we expand upon the method from [8, Sections 17–18], and use the following proposition.

**Proposition 5.2.** *Let  $(G, \gamma)$  be a reflection-(2, 2) graph. Then a generic direction network on the symmetric lift  $(\tilde{G}, \varphi)$  of  $(G, \gamma)$  has only collapsed realizations.*

Since the proof of Proposition 5.2 requires a detailed construction, we first show how it implies Proposition 5.1.

**5.2. Proof that Proposition 5.2 implies Proposition 5.1** Let  $(G, \gamma)$  be a Ross graph, and assign directions  $\mathbf{d}$  to the edges of  $G$  such that, for any extension  $(G + ij, \gamma)$  of  $(G, \gamma)$  to a reflection-(2, 2) graph as in Proposition 2.4,  $\mathbf{d}$  can be extended to a set of directions that is generic in the sense of Proposition 5.2. This is possible because there are a finite number of such extensions.

For this choice of  $\mathbf{d}$ , the realization space of the direction network  $(\tilde{G}, \varphi, \mathbf{d})$  is 2-dimensional. Since solutions to (11) may be scaled or translated in the vertical direction, all solutions to  $(\tilde{G}, \varphi, \mathbf{d})$  are related by scaling and translation. It then follows that a pair of vertices in the fibers over  $i$  and  $j$  are either distinct from each other in all non-zero solutions to (11) or always coincide. In the latter case, adding the edge  $ij$  with any direction does not change the dimension of the solution space, no matter what direction we assign to it. It then follows that the solution spaces of generic direction networks on  $(\tilde{G}, \varphi, \mathbf{d})$  and  $(\widetilde{G + ij}, \varphi, \mathbf{d})$  have the same dimension, which is a contradiction by Proposition 5.2.

For the opposite direction, suppose  $(G, \gamma)$  is not a Ross-graph. A proof similar to that in Section 3.3 applies. If  $m < 2n - 2$ , then dimension counting tells us that the space of realizations cannot be unique up to translation and scaling. If  $m \geq 2n - 2$ , then  $(G, \gamma)$  has one of two types of circuits, either a reflection-(2, 2) subgraph or an (2, 2)-subgraph with trivial  $\rho$ -image. In the former case, we are done by Proposition 5.2, and in the latter case, the same proof from Section 3.3 applies.  $\square$

**5.3. Proof of Proposition 5.2** It is sufficient to construct a specific set of directions with this property. The rest of the proof gives such a construction and verifies that all the solutions are collapsed. Let  $(G, \gamma)$  be a reflection-(2, 2) graph associated to  $(\tilde{G}, \varphi)$ .

**Combinatorial decomposition** We apply Proposition 2.9 to decompose  $(G, \gamma)$  into a spanning tree  $T$  with all colors the identity and a reflection-(1, 1) graph  $X$ . For now, we further assume that  $X$  is connected.

**Assigning directions** Let  $\mathbf{v}$  be a direction vector that is not horizontal or vertical. For each edge  $ij \in T$ , set  $\mathbf{d}_{ij} = \mathbf{v}$ . Assign all the edges of  $X$  the vertical direction. Denote by  $\mathbf{d}$  this assignment of directions.

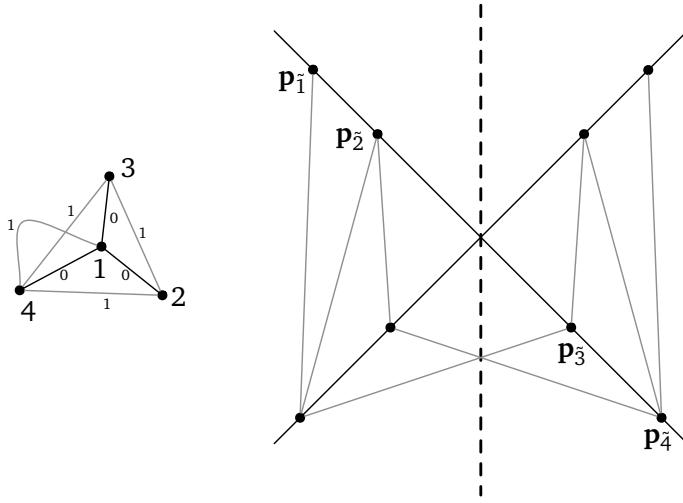


Figure 2: Schematic for proof of Proposition 5.2 and Lemma 5.4: the  $y$ -axis is shown as a dashed line. The corresponding colored graph is depicted on the left with the tree indicated by black lines and the reflection-(1,1) graph indicated by the gray lines. The right-hand figure indicates a realization where only the directions on the tree are enforced. If the gray lines are then forced to be vertical, the entire framework collapses to the  $y$ -axis. If the gray lines are forced to be horizontal, the framework takes on the form in Figure 3.

**All realizations are collapsed** We now show that the only realizations of  $(\tilde{G}, \varphi, \mathbf{d})$  have all vertices on top of each other. By Proposition 2.10,  $T$  lifts to two copies of itself, in  $\tilde{G}$ . It then follows from the connectivity of  $T$  and the construction of  $\mathbf{d}$  that, in any realization, there is a line  $L$  with direction  $\mathbf{v}$  such that every vertex of  $\tilde{G}$  must lie on  $L$  or its reflection. Since the vertical direction is preserved by reflection, the connectivity of the lift of  $X$ , again from Proposition 2.10, implies that every vertex of  $\tilde{G}$  lies on a single vertical line, which must be the  $y$ -axis by reflective symmetry.

Thus, in any realization of  $(\tilde{G}, \varphi, \mathbf{d})$  all the vertices lie at the intersection of  $L$ , the reflection of  $L$  through the  $y$ -axis and the  $y$ -axis itself. This is a single point, as desired. Figure 2 shows a schematic of this argument.

**$X$  does not need to be connected** Finally, we can remove the assumption that  $X$  was connected by repeating the argument for each connected component of  $X$  separately.  $\square$

**5.4. Special pairs for Ross-circuits** The full Theorem 4 will reduce to the case of a Ross-circuit.

**Proposition 5.3.** *Let  $(G, \gamma)$  be a Ross circuit with lift  $(\tilde{G}, \varphi)$ . Then there is an edge  $i'j'$  with nonzero color such that, for a generic direction network  $(\tilde{G}', \varphi, \mathbf{d})$  with colored graph  $(G - i'j', \gamma)$ :*

- *For all faithful realizations of  $(\tilde{G}', \varphi, \mathbf{d})$ , we have that  $p_{\tilde{j}'_1} - p_{\tilde{i}'_0}$  is a non-zero vector with direction independent of the realization. In particular,  $(\tilde{G}', \varphi, \mathbf{d})$  induces a well-defined direction on the edge  $i'j'$ , which extends to an assignment of directions to the edges of  $G$ .*
- *The direction networks  $(\tilde{G}, \varphi, \mathbf{d})$  and  $(\tilde{G}, \varphi, (\mathbf{d}^\perp))$  are a special pair.*

Before giving the proof, we describe the idea. We are after sets of directions that lead to faithful realizations of Ross-circuits. By Proposition 5.2, these directions must be non-generic. A natural way to obtain such a set of directions is to discard an edge  $ij$  from the colored quotient graph, apply Proposition 5.1 to obtain a generic set of directions  $\mathbf{d}'$  with a strongly faithful realization  $\tilde{G}'(\mathbf{p})$ , and then simply set the directions on the edges in the fiber over  $ij$  to be the difference vectors between the points.

Proposition 5.1 tells us that this procedure induces a well-defined direction for the edge  $ij$ , allowing us to extend  $\mathbf{d}'$  to  $\mathbf{d}$  in a controlled way. However, it does *not* tell us that rank of  $(\tilde{G}, \varphi, \mathbf{d})$  will rise when the directions are turned by angle  $\pi/2$ , and this seems hard to do directly. Instead, we construct a set of directions  $\mathbf{d}$  so that  $(\tilde{G}, \varphi, \mathbf{d})$  is rank deficient and has realizations where  $\mathbf{p}_i \neq \mathbf{p}_j$ , and  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  is generic. Then we make a perturbation argument to show the existence of a special pair.

The construction we use is, essentially, the one used in the proof of Proposition 5.2 but turned through angle  $\pi/2$ . The key geometric insight is that horizontal edge directions are preserved by the reflection, so the “gadget” of a line and its reflection crossing on the  $y$ -axis, as in Figure 2, degenerates to just a single line.

**5.5. Proof of Proposition 5.3** Let  $(G, \gamma)$  be a Ross-circuit; recall that this implies that  $(G, \gamma)$  is a reflection-Laman graph.

**Combinatorial decomposition** We decompose  $(G, \gamma)$  into a spanning tree  $T$  and a reflection-(1, 1) graph  $X$  as in Proposition 2.10. In particular, we again have all edges in  $T$  colored by the identity. For now, we *assume that  $X$  is connected*, and we fix  $i'j'$  to be an edge that is on the cycle in  $X$  with  $\gamma_{i'j'} \neq 0$ ; such an edge must exist by the hypothesis that  $X$  is reflection-(1, 1). Let  $G' = G \setminus i'j'$ . Furthermore, let  $\tilde{T}_0$  and  $\tilde{T}_1$  be the two connected components of the lift of  $T$ . For a vertex  $i \in G$ , the lift  $\tilde{i}$  lies in  $\tilde{T}_j$ . We similarly denote the lifts of  $i'$  and  $j'$  by  $\tilde{i}'_0, \tilde{i}'_1$  and  $\tilde{j}'_0, \tilde{j}'_1$ .

**Assigning directions** The assignment of directions is as follows: to the edges of  $T$ , we assign a direction  $v$  that is neither vertical nor horizontal. To the edges of  $X$  we assign the horizontal direction. Define the resulting direction network to be  $(\tilde{G}, \varphi, \mathbf{d})$ , and the direction network induced on the lift of  $G'$  to be  $(\tilde{G}', \varphi, \mathbf{d})$ .

**The realization space of  $(\tilde{G}, \varphi, \mathbf{d})$**  Figure 2 and Figure 3 contain a schematic picture of the arguments that follow.

**Lemma 5.4.** *The realization space of  $(\tilde{G}, \varphi, \mathbf{d})$  is 2-dimensional and parameterized by exactly one representative in the fiber over the vertex  $i$  selected above.*

*Proof.* In a manner similar to the proof of Proposition 5.2, the directions on the edges of  $T$  force every vertex to lie either on a line  $L$  in the direction  $\mathbf{v}$  or its reflection. Since the lift of  $X$  is connected, we further conclude that all the vertices lie on a single horizontal line. Thus, all the points  $\mathbf{p}_{\tilde{j}_0}$  are at the intersection of the same horizontal line and  $L$  or its reflection. These determine the locations of the  $\mathbf{p}_{\tilde{j}_1}$ , so the realization space is parameterized by the location of  $\mathbf{p}_{\tilde{i}'_0}$ .  $\square$

Inspecting the argument more closely, we find that:

**Lemma 5.5.** *In any realization  $\tilde{G}(\mathbf{p})$  of  $(\tilde{G}, \varphi, \mathbf{d})$ , all the  $\mathbf{p}_{\tilde{j}_0}$  are equal and all the  $\mathbf{p}_{\tilde{j}_1}$  are equal.*

*Proof.* Because the colors on the edges of  $T$  are all zero, it lifts to two copies of itself, one of which spans the vertex set  $\{\tilde{j}_0 : j \in V(G)\}$  and one which spans  $\{\tilde{j}_1 : j \in V(G)\}$ . It follows that in a realization, we have all the  $\mathbf{p}_{\tilde{j}_0}$  on  $L$  and the  $\mathbf{p}_{\tilde{j}_1}$  on the reflection of  $L$ .  $\square$

In particular, because the color  $\gamma_{i'j'}$  on the edge  $i'j'$  is 1, we obtain the following.

**Lemma 5.6.** *The realization space of  $(\tilde{G}, \varphi, \mathbf{d})$  contains points where the fiber over the edge  $i'j'$  is not collapsed.*

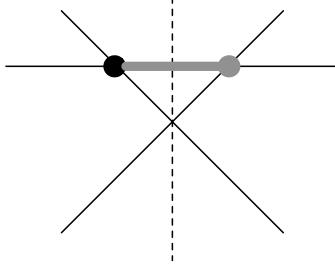


Figure 3: Schematic of the proof of Proposition 5.3: the  $y$ -axis is shown as a dashed line. The directions on the edges of the lift of the tree  $T$  force all the vertices to be on one of the two lines meeting at the  $y$ -axis. The horizontal directions on the connected reflection-(1, 1) graph  $X$  force the point  $\mathbf{p}_{\tilde{j}_0}$  to be at the intersection marked by the black dot and  $\mathbf{p}_{\tilde{j}_1}$  to be at the intersection marked by the gray one. The thick gray line indicates a thick mass of horizontal edges.

**The realization space of  $(\tilde{G}', \varphi, \mathbf{d})$**  The conclusion of Lemma 5.4 implies that the realization system for  $(\tilde{G}, \varphi, \mathbf{d})$  is rank deficient by one. Next we show that removing the edge  $i'j'$  results in a direction network that has full rank on the colored graph  $(G', \gamma)$ .

**Lemma 5.7.** *The realization space of  $(\tilde{G}, \varphi, \mathbf{d})$  is canonically identified with that of  $(\tilde{G}', \varphi, \mathbf{d})$ .*

*Proof.* In the proof of Lemma 5.4, it was not essential that  $X$  lifts to a connected subgraph of  $\tilde{G}$ . It was only required that  $X$  spans the vertices, and this is true of  $X - i'j'$ . Since the two lifts of a vertex must always lie on the same horizontal line in a realization, if any lift of  $i$  and any lift of  $j$  lie on the same horizontal line, then all lifts do. It is then easy to conclude all points lie on a single horizontal line.  $\square$

**The realization space of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$**  Next, we consider what happens when we turn all the directions by  $\pi/2$ .

**Lemma 5.8.** *The realization space of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has only collapsed solutions.*

*Proof.* This is exactly the construction used to prove Proposition 5.2.  $\square$

**Perturbing  $(\tilde{G}, \varphi, \mathbf{d})$**  To summarize what we have shown so far:

- (a)  $(\tilde{G}, \varphi, \mathbf{d})$  has a 2-dimensional realization space parameterized by  $\mathbf{p}_{\tilde{i}'_0}$  and identified with that of a full-rank direction network on the Ross graph  $(G', \gamma)$ .
- (b) There are points  $\tilde{G}(\mathbf{p})$  in this realization space where  $\mathbf{p}_{\tilde{i}'_0} \neq \mathbf{p}_{\tilde{j}'_1}$ .
- (c)  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has a 1-dimensional realization space containing only collapsed solutions.

What we have not shown is that the realization space of  $(\tilde{G}, \varphi, \mathbf{d})$  has *faithful* realizations, since the ones we constructed all have many coincident vertices. Proposition 5.1 will imply the rest of the theorem, provided that the above properties hold for any small perturbation of  $\mathbf{d}$ , since some small perturbation of *any* assignment of directions to the edges of  $(G', \gamma)$  has only faithful realizations.

**Lemma 5.9.** *Let  $\hat{\mathbf{d}}'$  be a perturbation of the directions  $\mathbf{d}$  on the edges of  $G'$  only. If  $\hat{\mathbf{d}}'$  is sufficiently close to  $\mathbf{d}|_{E(G')}$ , then there are realizations of the direction network  $(\tilde{G}', \varphi, \hat{\mathbf{d}}')$  such that the direction of  $\mathbf{p}_{\tilde{j}'_1} - \mathbf{p}_{\tilde{i}'_0}$  is nonzero and a small perturbation of  $\mathbf{d}_{ij}$ .*

*Proof.* The realization space is parameterized by  $\mathbf{p}_{\tilde{i}'_0}$  (for directions sufficiently close to  $\mathbf{d}'$ ), and so  $\mathbf{p}_{\tilde{j}'_1}$  varies continuously with the directions on the edges and  $\mathbf{p}_{\tilde{i}'_0}$ . Since there are realizations of  $(\tilde{G}', \varphi, \mathbf{d})$  with  $\mathbf{p}_{\tilde{i}'_0} \neq \mathbf{p}_{\tilde{j}'_1}$ , the Lemma follows.  $\square$

Lemma 5.9 implies that any sufficiently small perturbation of the directions assigned to the edges of  $G'$  gives a direction network that induces a well-defined direction on the edge  $i'j'$  which is itself a small perturbation of  $\mathbf{d}_{i'j'}$ . Since the ranks of  $(\tilde{G}', \varphi, \mathbf{d}')$  and  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  are stable under small perturbations, this implies that we can perturb  $\mathbf{d}$  to a  $\hat{\mathbf{d}}$  so that  $\hat{\mathbf{d}}|_{E(G')}$  is generic in the sense of Proposition 5.1, while preserving faithful realizability of  $(\tilde{G}, \varphi, \hat{\mathbf{d}})$  and full rank of the realization system for  $(\tilde{G}, \varphi, \hat{\mathbf{d}}^\perp)$ . The Proposition is proved for when  $X$  is connected.

**$X$  need not be connected** The proof is then complete once we remove the additional assumption that  $X$  was connected. Let  $X$  have connected components  $X_1, X_2, \dots, X_c$ . For each of the  $X_i$ , we can identify an edge  $(i'j')_k$  with the same properties as  $i'j'$  above.

Assign directions to the tree  $T$  as above. For  $X_1$ , we assign directions exactly as above. For each of the  $X_k$  with  $k \geq 2$ , we assign the edges of  $X_k \setminus (i'j')_k$  the horizontal direction and  $(i'j')_k$  a direction that is a small perturbation of horizontal.

With this assignment  $\mathbf{d}$  we see that for any realization of  $(\tilde{G}, \varphi, \mathbf{d})$ , each of the  $X_k$ , for  $k \geq 2$  is realized as completely collapsed to a single point at the intersection of the line  $L$  and the  $y$ -axis. Moreover, in the direction network on  $\mathbf{d}^\perp$ , the directions on these  $X_i$  are a small perturbation of the ones used on  $X$  in the proof of Proposition 5.2. From this it follows that any realization of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  is completely collapsed and hence full rank.

We now see that this new set of directions has properties (a), (b), and (c) above required for the perturbation argument. Since that argument makes no reference to the decomposition, it applies verbatim to the case where  $X$  is disconnected.  $\square$

## 5.6. Proof of Theorem 4

The easier direction to check is necessity.

**The Maxwell direction** If  $(G, \gamma)$  is not reflection-Laman, then it contains either a Laman-circuit with trivial  $\rho$ -image, or a violation of  $(2, 1)$ -sparsity. A violation of  $(2, 1)$ -sparsity implies that the realization system (11) of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has a dependency, since the realization space is always at least 1-dimensional.

Suppose instead there is a Laman-circuit  $G'$  with trivial  $\rho$ -image. Then any direction network on  $(G', \gamma)$  is equivalent to a direction network on the (finite, uncolored, non-symmetric) graph  $G'$ . (The lift  $\tilde{G}'$  is two mirror images of  $G'$ .) In this case, similar to Proposition 4.1,  $(G', \gamma, \mathbf{d})$  and  $(G', \gamma, \mathbf{d}^\perp)$  have the same rank. Thus if  $(G, \gamma, \mathbf{d}^\perp)$  and hence  $(G', \gamma, \mathbf{d}^\perp)$  has only collapsed realizations, so does  $(G', \gamma, \mathbf{d})$  in which case  $(G, \gamma, \mathbf{d})$  has no faithful realization.

**The Laman direction** Now let  $(G, \gamma)$  be a reflection-Laman graph and let  $(G', \gamma)$  be a Ross-basis of  $(G, \gamma)$ . For any edge  $ij \notin G'$ , adding it to  $G'$  induces a Ross-circuit which contains some edge  $i'j'$  having the property specified in Proposition 5.3. Note that  $G' - ij + i'j'$  is again a Ross-basis. We therefore can assume (after edge-swapping in this manner) for all  $ij \notin G'$  that  $ij$  has the property from Proposition 5.3 in the Ross-circuit it induces.

We assign directions  $\mathbf{d}'$  to the edges of  $G'$  such that:

- The directions on each of the intersections of the Ross-circuits with  $G'$  are generic in the sense of Proposition 5.3.
- The directions on the edges of  $G'$  that remain in the reduced graph  $(G^*, \gamma)$  are perpendicular to an assignment of directions on  $G^*$  that is generic in the sense of Proposition 5.2.
- The directions on the edges of  $G'$  are generic in the sense of Proposition 5.1.

This is possible because the set of disallowed directions is the union of a finite number of proper algebraic subsets in the space of direction assignments. Extend to directions  $\mathbf{d}$  on  $G$  by assigning directions to the remaining edges as specified by Proposition 5.3. By construction, we know that:

**Lemma 5.10.** *The direction network  $(\tilde{G}, \varphi, \mathbf{d})$  has faithful realizations.*

*Proof.* The realization space is identified with that of  $(\tilde{G}', \varphi, \mathbf{d}')$ , and  $\mathbf{d}'$  is chosen so that Proposition 5.1 applies.  $\square$

**Lemma 5.11.** *In any realization of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$ , the Ross-circuits are realized with all their vertices coincident and on the  $y$ -axis.*

*Proof.* This follows from how we chose  $\mathbf{d}$  and Proposition 5.3.  $\square$

As a consequence of Lemma 5.11, and the fact that we picked  $\mathbf{d}$  so that  $\mathbf{d}^\perp$  extends to a generic assignment of directions  $(\mathbf{d}^*)^\perp$  on the reduced graph  $(G^*, \gamma)$  we have:

**Lemma 5.12.** *The realization space of  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  is identified with that of  $(\tilde{G}^*, \varphi, (\mathbf{d}^*)^\perp)$  which, furthermore, contains only collapsed solutions.*

Observe that a direction network for a single self-loop (colored 1) with a generic direction only has solutions where vertices are collapsed and on the  $y$ -axis. Consequently, replacing a Ross-circuit with a single vertex and a self-loop yields isomorphic realization spaces. Since the reduced graph is reflection-(2, 2) by Proposition 2.6 and the directions assigned to its edges were chosen generically for Proposition 5.2, that  $(\tilde{G}, \varphi, \mathbf{d}^\perp)$  has only collapsed solutions follows. Thus, we have exhibited a special pair, completing the proof.  $\square$

**Remark** It can be seen that the realization space of a direction network as supplied by Theorem 4 has at least one degree of freedom for each edge that is not in a Ross basis. Thus, the statement cannot be improved to, e.g., a unique realization up to translation and scale.

## 6. Infinitesimal rigidity of reflection frameworks

Let  $(\tilde{G}, \varphi, \ell)$  be a reflection framework and let  $(G, \gamma)$  be the quotient graph. The configuration space, which is the set of solutions to the quadratic system (1)–(2) is canonically identified with the solutions to:

$$\|\Phi(\gamma_{ij}) \cdot \mathbf{p}_j - \mathbf{p}_i\|^2 = \ell_{ij}^2 \quad \text{for all edges } ij \in E(G) \quad (14)$$

where  $\Phi$  acts on the plane by reflection through the  $y$ -axis. (That “pinning down”  $\Phi$  does not affect the theory is straightforward from the definition of the configuration space: it simply removes rotation and translation in the  $x$ -direction from the set of trivial motions.)

**6.1. Infinitesimal rigidity** Computing the formal differential of (14), we obtain the system

$$\langle \Phi(\gamma_{ij}) \cdot \mathbf{p}_j - \mathbf{p}_i, \mathbf{v}_j - \mathbf{v}_i \rangle = 0 \quad \text{for all edges } ij \in E(G) \quad (15)$$

where the unknowns are the *velocity vectors*  $\mathbf{v}_i$ . A standard kind of result (cf. [1]) is the following.

**Proposition 6.1.** *Let  $\tilde{G}(\mathbf{p}, \Phi)$  be a realization of an abstract framework  $(\tilde{G}, \varphi, \ell)$ . If the corank of the system (15) is one, then  $\tilde{G}(\mathbf{p})$  is rigid.*

Thus, we define a realization to be *infinitesimally rigid* if the system (15) has maximal rank, and *minimally infinitesimally rigid* if it is infinitesimally rigid but ceases to be so after removing any edge from the colored quotient graph.

By definition, infinitesimal rigidity is defined by a polynomial condition in the coordinates of the points  $\mathbf{p}_i$ , so it is a generic property associated with the colored graph  $(G, \gamma)$ .

**6.2. Relation to direction networks** Here is the core of the direction network method for reflection frameworks: we can understand the rank of (15) in terms of a direction network.

**Proposition 6.2.** *Let  $\tilde{G}(\mathbf{p}, \Phi)$  be a realization of a reflection framework with  $\Phi$  acting by reflection through the  $y$ -axis. Define the direction  $\mathbf{d}_{ij}$  to be  $\Phi(\gamma_{ij}) \cdot \mathbf{p}_j - \mathbf{p}_i$ . Then the rank of (15) is equal to that of (11) for the direction network  $(G, \gamma, \mathbf{d}^\perp)$ .*

*Proof.* Exchange the roles of  $\mathbf{v}_i$  and  $\mathbf{p}_i$  in (15). □

**6.3. Proof of Theorem 2** The, more difficult, “Laman direction” of the Main Theorem follows immediately from Theorem 4 and Proposition 6.2: given a reflection-Laman graph Theorem 4 produces a realization with no coincident endpoints and a certificate that (15) has corank one. The “Maxwell direction” follows from a similar argument as that for Theorem 1. □

**Remark** The statement of Proposition 6.2 is *exactly the same* as the analogous statement for orientation-preserving cases of this theory. What is different is that, for reflection frameworks, the rank of  $(G, \gamma, \mathbf{d}^\perp)$  is *not*, the same as that of  $(G, \gamma, \mathbf{d})$ . By Proposition 5.2, the set of directions arising as the difference vectors from point sets are *always non-generic* on reflection-Laman graphs, so we are forced to introduce the notion of a special pair.

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